Estimation in uncertain switched systems using a bank of interval observers: local vs glocal approach ⋆

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Abstract: This paper discusses some issues related with the design of a bank of interval observers for uncertain switched systems, in which several sources of uncertainty are considered: parametric uncertainties, unknown disturbances, measurement noise, and unknown switching signal. More specifically, this paper focuses on analyzing the interval estimation accuracy when changes of active mode induce non-positivity of the interval state estimation errors. In particular, it is shown that by combining two types of interval observers, referred to as *local* and *global*, the accuracy and reliability of the estimation can be improved. The properties of the obtained so-called *glocal* observer are investigated and illustrated by means of numerical simulations.

Keywords: Interval observers, switched systems, mode identification, uncertain systems.

1. INTRODUCTION

It is a well known fact that, due to uncertainties coming either from external disturbances or from the mismatch between the model and the real system, the classical state observers cannot converge to the real value of the state in general settings [Efimov et al., 2016, Wang et al., 2015]. For this reason, interval observers have been considered as an appealing alternative approach, since they can take into account the information that uncertainties and disturbances are bounded in some known sets, in order to compute the set of admissible values for the state at each instant of time [Efimov et al., 2012]. Research on interval observers is still a hot topic in present days, with several contributions appearing in the literature, concerning their integration with advanced control techniques [Oubabas et al., 2018], fault diagnosis [Rotondo et al., 2018b] and fault tolerant control [Rotondo et al., 2018a] problems.

On the other hand, switched systems [Daafouz et al., 2002] represent a special class of hybrid dynamics that is applied in several fields, such as control of chemical processes [Niu et al., 2015] and flight control systems [Sakthivel et al., 2016]. In the last few years, some results concerning state estimation in switched systems have appeared, see for instance Zhao et al. [2015], Ríos et al. [2015], Rios et al. [2014], where an important difference comes from whether the hypothesis on availability of the switching signal is made or not. Some works have considered the problem of interval state estimation in switched systems. For instance, Ethabet et al. [2018b,a] have proposed an interval observer design approach for continuous-time switched systems affected by unknown inputs. He and Xie [2016] have addressed control system design based on an interval observer for non-linear switched systems with Lipschitz non-linearities. Ifqir et al. [2017] have analyzed interval estimation accuracy and robustness with respect to unknown disturbances using $\mathcal{H}_\infty$ objective with pole placement constraints.

However, all the above works have been developed under the assumption that the switching signal that defines the active mode of the switched system is known. This assumption is not always true [Wang et al., 2018], and some research has addressed the issue of determining the active mode at any

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moment, using only the system's input/output data, see e.g. Lee et al. [2013], Hakem et al. [2016], Rotondo et al. [2017b,a]. In particular, Rotondo et al. [2017b,a] showed that this goal could be achieved using a multiple model adaptive architecture (MMAE) that relies on a set of *local* observers, each designed using one among the possible modes of the switched system.

With the future ultimate goal of extending the MMAE to the case where interval observers are used to estimate the switching signal while taking into account the effects of the different sources of uncertainty, this paper focuses on analyzing the interval estimation accuracy when changes of active mode induce non-positivity of the interval state estimation errors. In particular, the main contribution of this work is to show that the accuracy of the estimation can be improved by using a combination of two types of interval observers, referred to as *local* and *global*. The properties of the obtained observer, named *glocal* after Hara et al. [2015], are investigated and demonstrated by means of numerical simulations.

The paper is structured as follows. Section 2 presents some notation used in the paper, along with a useful lemma and the definition of cooperative system. Section 3 contains the theoretical results concerning the definition of the considered class of systems, and the structure of the local, global and glocal interval observers. Simulation results are shown and discussed in Section 4, whereas Section 5 presents the conclusions.

2. PRELIMINARIES

Denote by $\mathbb{R}$ and $\mathbb{N}$ ($\mathbb{N}_0$) the sets of real and natural numbers (with zero included), respectively, and $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$. For two vectors $x_1, x_2 \in \mathbb{R}^n$ or matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$, the relations $x_1 \leq x_2$ and $A_1 \leq A_2$ are understood elementwise. Given a matrix $A \in \mathbb{R}^{n\times n}$, let us define $A^+ = \max\{0, A\}$, with $\max$ understood elementwise, $A^- = A^+ - A$ and $|A| = A^+ + A^-$ (similarly for vectors).

Lemma 1. [Efimov et al., 2016] Let $\underline{A} \leq A \leq \overline{A}$ for some $\underline{A}$, $\overline{A}$, $A \in \mathbb{R}^{n\times n}$ and $\underline{x} \leq x \leq \overline{x}$ for $\underline{x}, \overline{x}, x \in \mathbb{R}^n$, then:

$$\begin{aligned}\underline{A}^+\underline{x}^+ - \overline{A}^+\underline{x}^- - \underline{A}^-\overline{x}^+ + \overline{A}^-\overline{x}^- &\leq Ax \\ &\leq \overline{A}^+\overline{x}^+ - \underline{A}^+\overline{x}^- - \overline{A}^-\underline{x}^+ + \underline{A}^-\underline{x}^-\end{aligned} \tag{1}$$

Let us define as $\mathbb{R}_+^{n\times n}$ the set of matrices $A \in \mathbb{R}^{n\times n}$ with nonnegative elements ($A \geq 0$). Then, any solution of the system:

$$x(t+1) = Ax(t) + \omega(t), \omega : \mathbb{N} \to \mathbb{R}_+^n, t \in \mathbb{N} \tag{2}$$

with $x(t) \in \mathbb{R}^n$ and $A \in \mathbb{R}_+^{n\times n}$, is elementwise nonnegative for all $t \geq 0$ if $x(0) \geq 0$. Such a system is referred to as *cooperative* (or *monotone*) [Hirsch and Smith, 2005].

3. THEORETICAL RESULTS

3.1 System definition

Let us consider a discrete-time uncertain switched system described by:

$$\begin{aligned}x(t+1) = &\left[A(\kappa(t)) + \Delta A(\kappa(t))\right]x(t) \\ &+ \left[B(\kappa(t)) + \Delta B(\kappa(t))\right]u(t) + d(t)\end{aligned} \tag{3}$$

$$y(t) = Cx(t) + v(t) \tag{4}$$

where $t \in \mathbb{N}_0$ stands for the sample, $x \in \mathbb{R}^{n_x}$ is the state, $y \in \mathbb{R}_{n_y}$ is the output, $u \in \mathbb{R}^{n_u}$ is the input, $\kappa(t) \in \{\kappa_1, \kappa_2, \dots, \kappa_J\}$ denotes the switching signal, while the signals $d \in \mathbb{R}^{n_x}$ and $v \in \mathbb{R}^{n_y}$ denote the exogenous disturbance and measurement noise, respectively, which are unknown. The matrix $C \in \mathbb{R}^{n_y\times n_x}$ and the matrix functions $A(\kappa(t)) \in \mathbb{R}^{n_x\times n_x}$, $B(\kappa(t)) \in \mathbb{R}^{n_x\times n_u}$ are assumed to be known (although the instantaneous value of $\kappa(t)$ is not), whereas the matrix functions $\Delta A(\kappa(t)) \in \mathbb{R}^{n_x\times n_x}$, $\Delta B(\kappa(t)) \in \mathbb{R}^{n_x\times n_u}$ are unknown. The following assumption will be used in further developments.

Assumption 1. $\underline{d}(t) \leq d(t) \leq \overline{d}(t)$ and $|v(t)| \leq V$ for all $t \in \mathbb{N}_0$ and for known $\underline{d}(t), \overline{d}(t) \in \mathbb{R}^{n_x}$ and $V \in \mathbb{R}_+$. Moreover:

$$\underline{\Delta A}(\kappa(t)) \leq \Delta A(\kappa(t)) \leq \overline{\Delta A}(\kappa(t)) \tag{5}$$

$$\underline{\Delta B}(\kappa(t)) \leq \Delta B(\kappa(t)) \leq \overline{\Delta B}(\kappa(t)) \tag{6}$$

$$A_0 + \underline{\underline{\Delta A}} \leq A(\kappa(t)) + \Delta A(\kappa(t)) \leq A_0 + \overline{\overline{\Delta A}} \tag{7}$$

$$B_0 + \underline{\underline{\Delta B}} \leq B(\kappa(t)) + \Delta B(\kappa(t)) \leq B_0 + \overline{\overline{\Delta B}} \tag{8}$$

for some known matrix functions:

$$\begin{aligned}\underline{\Delta A}(\cdot), \overline{\Delta A}(\cdot) \in \mathbb{R}^{n_x\times n_x} &: \underline{\Delta A}(\cdot) \leq 0,\ \overline{\Delta A}(\cdot) \geq 0 \\ \underline{\Delta B}(\cdot), \overline{\Delta B}(\cdot) \in \mathbb{R}^{n_x\times n_u} &: \underline{\Delta B}(\cdot) \leq 0,\ \overline{\Delta B}(\cdot) \geq 0\end{aligned}$$

and some known matrices $A_0 \in \mathbb{R}^{n_x\times n_x}$, $B_0 \in \mathbb{R}^{n_x\times n_u}$ and:

$$\begin{aligned}\underline{\underline{\Delta A}}, \overline{\overline{\Delta A}} \in \mathbb{R}^{n_x\times n_x} &: \underline{\underline{\Delta A}} \leq 0, \overline{\overline{\Delta A}} \geq 0 \\ \underline{\underline{\Delta B}}, \overline{\overline{\Delta B}} \in \mathbb{R}^{n_x\times n_u} &: \underline{\underline{\Delta B}} \leq 0, \overline{\overline{\Delta B}} \geq 0\end{aligned}$$

It is straightforward to show that if $\kappa(t) = \kappa_j$, the system (3)-(4) reduces to:

$$x(t+1) = (A_j + \Delta A_j)\,x(t) + (B_j + \Delta B_j)\,u(t) + d(t) \tag{9}$$

$$y(t) = Cx(t) + v(t) \tag{10}$$

with known matrices $A_j \triangleq A(\kappa_j), B_j \triangleq B(\kappa_j)$ and, according to Assumption 1, matrices $\Delta A_j \triangleq \Delta A(\kappa_j)$ and $\Delta B_j \triangleq \Delta B(\kappa_j)$ unknown but such that:

$$\begin{aligned}\underline{\Delta A_j} \leq \Delta A_j \leq \overline{\Delta A_j} \\ \underline{\Delta B_j} \leq \Delta B_j \leq \overline{\Delta B_j}\end{aligned}$$

for some known matrices:

$$\begin{aligned}\underline{\Delta A_j} \triangleq \underline{\Delta A}(\kappa_j) \in \mathbb{R}^{n_x\times n_x} &: \underline{\Delta A_j} \leq 0 \\ \overline{\Delta A_j} \triangleq \overline{\Delta A}(\kappa_j) \in \mathbb{R}^{n_x\times n_x} &: \overline{\Delta A_j} \geq 0 \\ \underline{\Delta B_j} \triangleq \underline{\Delta B}(\kappa_j) \in \mathbb{R}^{n_x\times n_u} &: \underline{\Delta B_j} \leq 0 \\ \overline{\Delta B_j} \triangleq \overline{\Delta B}(\kappa_j) \in \mathbb{R}^{n_x\times n_u} &: \overline{\Delta B_j} \geq 0\end{aligned}$$

3.2 Local interval observers

For each subsystem (9)-(10), it is possible to propose a local interval observer based on cooperativity of the interval estimation error dynamics [Efimov et al., 2016], as follows:

$$\begin{aligned}\underline{x_j}(t+1) = &\left[A_j - \underline{L_j}C\right]\underline{x_j}(t) + B_j u(t) + \underline{L_j}y(t) \\ &- |\underline{L_j}|V1_{n_y} + \underline{d}(t) + \underline{\Delta A_j}^+\underline{x_j}^+(t) - \overline{\Delta A_j}^+\underline{x_j}^-(t) \\ &- \underline{\Delta A_j}^-\overline{x_j}^+(t) + \overline{\Delta A_j}^-\overline{x_j}^-(t) + \underline{\Delta B_j}^+u^+(t) \\ &- \overline{\Delta B_j}^+u^-(t) - \underline{\Delta B_j}^-u^+(t) + \overline{\Delta B_j}^-u^-(t)\end{aligned} \tag{11}$$

$$
\begin{aligned}
\overline{x_j}(t+1) = {} & \left[A_j - \overline{L_j}C\right]\overline{x_j}(t) + B_j u(t) + \overline{L_j}y(t) \\
& + |\overline{L_j}|V1_{n_y} + \overline{d}(t) + \overline{\Delta A_j}^{+}\overline{x_j}^{+}(t) - \underline{\Delta A_j}^{+}\overline{x_j}^{-}(t) \\
& - \overline{\Delta A_j}^{-}\underline{x_j}^{+}(t) + \underline{\Delta A_j}^{-}\underline{x_j}^{-}(t) + \overline{\Delta B_j}^{+}u^{+}(t) \\
& - \underline{\Delta B_j}^{+}u^{-}(t) - \overline{\Delta B_j}^{-}u^{+}(t) + \underline{\Delta B_j}^{-}u^{-}(t)
\end{aligned} \tag{12}
$$

where $\underline{x_j}, \overline{x_j}$ are the lower and upper interval estimates of $x(t)$ for the j-th subsystem, $\underline{L_j}, \overline{L_j} \in \mathbb{R}^{n_x \times n_y}$ are the observer gains to be designed, and 1_{n_y} denotes the column vector of length n_y with all elements equal to 1 .

The following proposition gives some conditions under which the local interval observer provides an interval estimation of $x(t)$.

Proposition 1. Let Assumption 1 be satisfied, and:

$$A_j - \underline{L_j}C, A_j - \overline{L_j}C \in \mathbb{R}_+^{n_x \times n_x} \tag{13}$$

Then, for $t_0, t_f \in \mathbb{N}_0$, the relation:

$$\underline{x_j}(t) \leq x(t) \leq \overline{x_j}(t) \quad \forall t \in \{t_0, t_0+1, \ldots, t_0+t_f\} \tag{14}$$

holds provided that:

$$\kappa(t) = \kappa_j \qquad \forall t \in \{t_0, t_0+1, \ldots, t_0+t_f\} \tag{15}$$

$$\underline{x_j}(t_0) \leq x(t_0) \leq \overline{x_j}(t_0) \tag{16}$$

Proof. Consider the dynamics of the interval estimation errors $\underline{e_j}(t) = x(t) - \underline{x_j}(t)$ and $\overline{e_j}(t) = \overline{x_j}(t) - x(t)$:

$$\underline{e_j}(t+1) = \left[A_j - \underline{L_j}C\right]\underline{e_j}(t) + \sum_{i=1}^{4}\underline{w_j^i}(t) \tag{17}$$

$$\overline{e_j}(t+1) = \left[A_j - \overline{L_j}C\right]\overline{e_j}(t) + \sum_{i=1}^{4}\overline{w_j^i}(t) \tag{18}$$

where:

$$
\begin{aligned}
\underline{w_j^1}(t) = {} & \Delta A_j x(t) - \underline{\Delta A_j}^{+}\underline{x_j}^{+}(t) + \overline{\Delta A_j}^{+}\underline{x_j}^{-}(t) \\
& + \underline{\Delta A_j}^{-}\overline{x_j}^{+}(t) - \overline{\Delta A_j}^{-}\overline{x_j}^{-}(t)
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
\underline{w_j^2}(t) = {} & \Delta B_j u(t) - \underline{\Delta B_j}^{+}u^{+}(t) + \overline{\Delta B_j}^{+}u^{-}(t) \\
& + \underline{\Delta B_j}^{-}u^{+}(t) - \overline{\Delta B_j}^{-}u^{-}(t)
\end{aligned} \tag{20}
$$

$$\underline{w_j^3}(t) = d(t) - \underline{d}(t) \tag{21}$$

$$\underline{w_j^4}(t) = |\underline{L_j}|V1_{n_y} + v(t) \tag{22}$$

$$
\begin{aligned}
\overline{w_j^1}(t) = {} & \overline{\Delta A_j}^{+}\overline{x_j}^{+}(t) - \underline{\Delta A_j}^{+}\overline{x_j}^{-}(t) - \overline{\Delta A_j}^{-}\underline{x_j}^{+}(t) \\
& + \underline{\Delta A_j}^{-}\underline{x_j}^{-}(t) - \Delta A_j x(t)
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
\overline{w_j^2}(t) = {} & \overline{\Delta B_j}^{+}u^{+}(t) - \underline{\Delta B_j}^{+}u^{-}(t) - \overline{\Delta B_j}^{-}u^{+}(t) \\
& + \underline{\Delta B_j}^{-}u^{-}(t) - \Delta B_j u(t)
\end{aligned} \tag{24}
$$

$$\overline{w_j^3}(t) = \overline{d}(t) - d(t) \tag{25}$$

$$\overline{w_j^4}(t) = |\overline{L_j}|V1_{n_y} - v(t) \tag{26}$$

Hence, the dynamics for $\underline{e_j}(t)$ and $\overline{e_j}(t)$ is cooperative and (14) holds as long as $\underline{w_j^i}(t), \overline{w_j^i}(t) \geq 0 \, \forall i = 1,2,3,4$ and $\forall t \geq t_0$, which is true due to Assumption 1. □

Under a change of active mode, condition (16) might not hold, in such a case Lemma 1 could not be applied anymore in order to ensure that the signals $\underline{w_j^1}(t)$ and $\overline{w_j^1}(t)$ described by (19) and (23), respectively, remain non-negative. The remaining of this section will show how, by defining a global interval observer, and using the estimated global bounds to feed the local observers, it is possible to ensure that the interval estimation errors are fed by non-negative inputs even in cases where a change of active mode causes (16) not to hold anymore.

Remark 1. Note that Proposition 1, along with similar propositions presented in the remaining of the paper, focuses only on the interval estimation property, without considering boundedness of the observers' states. Linear matrix inequality (LMI)-based conditions for designing appropriate observer gains such that boundedness is achieved can be found in the available literature, see e.g. Efimov et al. [2016].

3.3 Global interval observer

It is possible to take into account the uncertainty about $\kappa(t)$ by means of a global interval observer that guarantees the state to be always within some estimated lower and upper bounds, in spite of the varyingness of $\kappa(t)$, as follows:

$$
\begin{aligned}
\underline{\underline{x}}(t+1) = {} & \left[A_0 - \underline{\underline{L_0}}C\right]\underline{\underline{x}}(t) + B_0 u(t) + \underline{\underline{L_0}}y(t) \\
& - |\underline{\underline{L_0}}|V1_{n_y} + \underline{d}(t) + \underline{\underline{\Delta A}}^{+}\underline{\underline{x}}^{+}(t) - \overline{\overline{\Delta A}}^{+}\underline{\underline{x}}^{-}(t) \\
& - \underline{\underline{\Delta A}}^{-}\overline{\overline{x}}^{+}(t) + \overline{\overline{\Delta A}}^{-}\overline{\overline{x}}^{-}(t) + \underline{\underline{\Delta B}}^{+}u^{+}(t) \\
& - \overline{\overline{\Delta B}}^{+}u^{-}(t) - \underline{\underline{\Delta B}}^{-}u^{+}(t) + \overline{\overline{\Delta B}}^{-}u^{-}(t)
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
\overline{\overline{x}}(t+1) = {} & \left[A_0 - \overline{\overline{L_0}}C\right]\overline{\overline{x}}(t) + B_0 u(t) + \overline{\overline{L_0}}y(t) \\
& + |\overline{\overline{L_0}}|V1_{n_y} + \overline{d}(t) + \overline{\overline{\Delta A}}^{+}\overline{\overline{x}}^{+}(t) - \underline{\underline{\Delta A}}^{+}\overline{\overline{x}}^{-}(t) \\
& - \overline{\overline{\Delta A}}^{-}\underline{\underline{x}}^{+}(t) + \underline{\underline{\Delta A}}^{-}\underline{\underline{x}}^{-}(t) + \overline{\overline{\Delta B}}^{+}u^{+}(t) \\
& - \underline{\underline{\Delta B}}^{+}u^{-}(t) - \overline{\overline{\Delta B}}^{-}u^{+}(t) + \underline{\underline{\Delta B}}^{-}u^{-}(t)
\end{aligned} \tag{28}
$$

Proposition 2. Let Assumption 1 be satisfied, and:

$$A_0 - \underline{\underline{L_0}}C, A_0 - \overline{\overline{L_0}}C \in \mathbb{R}_+^{n_x \times n_x} \tag{29}$$

Then, for $t_0, t_f \in \mathbb{N}_0$, the relation:

$$\underline{\underline{x}}(t) \leq x(t) \leq \overline{\overline{x}}(t) \qquad \forall t \in \{t_0, t_0+1, \ldots, t_0+t_f\} \tag{30}$$

holds provided that:

$$\underline{\underline{x}}(t_0) \leq x(t_0) \leq \overline{\overline{x}}(t_0) \tag{31}$$

Proof. Due to Assumption 1, in particular Eqs. (7)-(8), the following is true:

$$A(\kappa(t)) + \Delta A(\kappa(t)) = A_0 + \nabla A(\kappa(t)) \tag{32}$$

$$B(\kappa(t)) + \Delta B(\kappa(t)) = B_0 + \nabla B(\kappa(t)) \tag{33}$$

for some $\nabla A(\kappa(t)), \nabla B(\kappa(t))$ such that:

$$\underline{\underline{\Delta A}} \leq \nabla A(\kappa(t)) \leq \overline{\overline{\Delta A}}$$

$$\underline{\underline{\Delta B}} \leq \nabla B(\kappa(t)) \leq \overline{\overline{\Delta B}}$$

Hence, the dynamics of the interval estimation errors $\underline{\underline{e}}(t+1) = x(t) - \underline{\underline{x}}(t)$ and $\overline{\overline{e}}(t) = \overline{\overline{x}}(t) - x(t)$ are described by:

$$\underline{\underline{e}}(t+1) = \left[A_0 - \underline{\underline{L_0}}C\right]\underline{\underline{e}}(t) + \sum_{i=1}^{4}\underline{\underline{w^i}}(t) \tag{34}$$

$$\overline{\overline{e}}(t+1) = \left[A_0 - \overline{\overline{L_0}}C\right]\overline{\overline{e}}(t) + \sum_{i=1}^{4}\overline{\overline{w^i}}(t) \tag{35}$$

where $\underline{\underline{w^i}}(t)$, $\overline{\overline{w^i}}(t)$ can be obtained from $\underline{w_j^i}(t), \overline{w_j^i}(t)$ in (19)-(26) by replacing $\Delta A_j \to \nabla A(\kappa(t))$, $\underline{\Delta A_j} \to \underline{\underline{\Delta A}}$, $\overline{\Delta A_j} \to$

$\overline{\overline{\Delta A}}$, $\underline{x_j} \to \underline{\underline{x}}$, $\overline{x_j} \to \overline{\overline{x}}$, $\Delta B_j \to \nabla B(\kappa(t))$, $\underline{\Delta B_j} \to \underline{\underline{\Delta B}}$, $\overline{\Delta B_j} \to \overline{\overline{\Delta B}}$, $\underline{L_j} \to \underline{\underline{L_0}}$, $\overline{L_j} \to \overline{\overline{L_0}}$. The remaining of the proof follows a similar reasoning as in the one of Proposition 1. □

3.4 Glocal interval observers

The information provided by the global interval observer can be fed locally to interval observers for different subsystems (9)-(10), as follows:

$$\begin{aligned}\underline{\underline{x_j}}(t+1) = {} & \left[A_j - \underline{\underline{L_j}}C\right]\underline{\underline{x_j}}(t) + B_j u(t) + \underline{\underline{L_j}}y(t) \\ & - |\underline{\underline{L_j}}|V1_{n_y} + \underline{d}(t) + \underline{\Delta A_j}^{+}\underline{\underline{x}}^{+}(t) - \overline{\Delta A_j}^{+}\underline{\underline{x}}^{-}(t) \\ & - \underline{\Delta A_j}^{-}\overline{\overline{x}}^{+}(t) + \overline{\Delta A_j}^{-}\overline{\overline{x}}^{-}(t) + \underline{\Delta B_j}^{+}u^{+}(t) \\ & - \overline{\Delta B_j}^{+}u^{-}(t) - \underline{\Delta B_j}^{-}u^{+}(t) + \overline{\Delta B_j}^{-}u^{-}(t)\end{aligned} \quad (36)$$

$$\begin{aligned}\overline{\overline{x_j}}(t+1) = {} & \left[A_j - \overline{\overline{L_j}}C\right]\overline{\overline{x_j}}(t) + B_j u(t) + \overline{\overline{L_j}}y(t) \\ & + |\overline{\overline{L_j}}|V1_{n_y} + \overline{d}(t) + \overline{\Delta A_j}^{+}\overline{\overline{x}}^{+}(t) - \underline{\Delta A_j}^{+}\overline{\overline{x}}^{-}(t) \\ & - \overline{\Delta A_j}^{-}\underline{\underline{x}}^{+}(t) + \underline{\Delta A_j}^{-}\underline{\underline{x}}^{-}(t) + \overline{\Delta B_j}^{+}u^{+}(t) \\ & - \underline{\Delta B_j}^{+}u^{-}(t) - \overline{\Delta B_j}^{-}u^{+}(t) + \underline{\Delta B_j}^{-}u^{-}(t)\end{aligned} \quad (37)$$

We will refer to an observer in the form (36)-(37) as *glocal*. The advantage of considering a glocal observer lies in the fact that thanks to the introduced feed, it can be guaranteed that the dynamics of the estimation errors $\underline{\underline{e_j}}(t) = x(t) - \underline{\underline{x_j}}(t)$ and $\overline{\overline{e_j}}(t) = \overline{\overline{x_j}}(t) - x(t)$ will be excited by non-negative inputs even when changes of active mode induce their non-positivity at some sample.

Proposition 3. Let Assumption 1 be satisfied, and for $t_0, t_f \in \mathbb{N}_0$:

$$A_j - \underline{\underline{L_j}}C, A_j - \overline{\overline{L_j}}C \in \mathbb{R}_+^{n_x \times n_x} \quad (38)$$

$$\kappa(t) = \kappa_j \qquad \forall t \in \{t_0, t_0+1, \ldots, t_0+t_f\} \quad (39)$$

Then, the relation:

$$\underline{\underline{x_j}}(t) \le x(t) \le \overline{\overline{x_j}}(t) \qquad \forall t \in \{t_0, t_0+1, \ldots, t_0+t_f\} \quad (40)$$

holds provided that:

$$\underline{\underline{x_j}}(t_0) \le x(t_0) \le \overline{\overline{x_j}}(t_0) \quad (41)$$

Moreover, let us define:

$$\underline{\underline{e_j^w}}(t_0+N) \triangleq \sum_{k=0}^{N-1}\sum_{i=1}^{4}\left[A_j - \underline{\underline{L_j}}C\right]^{N-k-1}\underline{\underline{w_j^i}}(t_0+k) \quad (42)$$

$$\overline{\overline{e_j^w}}(t_0+N) \triangleq \sum_{k=0}^{N-1}\sum_{i=1}^{4}\left[A_j - \overline{\overline{L_j}}C\right]^{N-k-1}\overline{\overline{w_j^i}}(t_0+k) \quad (43)$$

If $x(t_0) \le \underline{\underline{x_j}}(t_0)$ or $x(t_0) \ge \overline{\overline{x_j}}(t_0)$ and there exists $N \in \mathbb{N}$ such that:

$$\underline{\underline{e_j^w}}(t_0+N) \ge -\left[A_j - \underline{\underline{L_j}}C\right]^N \underline{\underline{e_j}}(t_0) \quad (44)$$

or:

$$\overline{\overline{e_j^w}}(t_0+N) \ge -\left[A_j - \overline{\overline{L_j}}C\right]^N \overline{\overline{e_j}}(t_0) \quad (45)$$

respectively [1], then:

$$\underline{x_j}(t_0+N) \le x(t_0+N) \le \overline{\overline{x_j}}(t_0+N) \quad (46)$$

[1] Conditions (44)-(45) can be interpreted as requiring that the signals $\underline{\underline{w_j^i}}(t_0 + k)$ and $\overline{\overline{w_j^i}}(t_0+k)$, $i = 1, 2, 3, 4$, are sufficiently exciting in the time interval $t_0, \ldots, t_0 + N - 1$.

Proof: The dynamics of the interval estimation errors $\underline{\underline{e_j}}(t)$ and $\overline{\overline{e_j}}(t)$ are given by:

$$\underline{\underline{e_j}}(t+1) = \left[A_j - \underline{\underline{L_j}}C\right]\underline{\underline{e_j}}(t) + \sum_{i=1}^{4}\underline{\underline{w_j^i}}(t) \quad (47)$$

$$\overline{\overline{e_j}}(t+1) = \left[A_j - \overline{\overline{L_j}}C\right]\overline{\overline{e_j}}(t) + \sum_{i=1}^{4}\overline{\overline{w_j^i}}(t) \quad (48)$$

where $\underline{\underline{w_j^i}}(t)$, $\overline{\overline{w_j^i}}(t)$ can be obtained from $\underline{w_j^i}(t), \overline{w_j^i}(t)$ in (19)-(26) by replacing $\underline{x_j} \to \underline{\underline{x}}$, $\overline{x_j} \to \overline{\overline{x}}$, $\underline{L_j} \to \underline{\underline{L_j}}$, $\overline{L_j} \to \overline{\overline{L_j}}$. The proof that (41) implies (40) follows a similar reasoning as in the proof of Proposition 1. Let us notice that the response of (47)-(48) is given by:

$$\underline{\underline{e_j}}(t_0+N) = \left[A_j - \underline{\underline{L_j}}C\right]^N \underline{\underline{e_j}}(t_0) + \underline{\underline{e_j^w}}(t_0+N) \quad (49)$$

$$\overline{\overline{e_j}}(t_0+N) = \left[A_j - \overline{\overline{L_j}}C\right]^N \overline{\overline{e_j}}(t_0) + \overline{\overline{e_j^w}}(t_0+N) \quad (50)$$

Then, it is straightforward that (46) follows from (44)-(45). □

Remark 2. The last result of Proposition 3 implies that in the cases when the instants of commutation are unknown, and we cannot reinitialize the interval observer (36)-(37) properly for the correct value of κ, if the inputs are sufficiently rich, then the glocal interval observer will start to produce correct state bounds after some transient. In fact, if the gains $\underline{\underline{L_j}}$ and $\overline{\overline{L_j}}$ are designed such that the matrices $A_j - \underline{\underline{L_j}}C$ and $A_j - \overline{\overline{L_j}}C$ are Schur stable, then the terms corresponding to the free responses from initial conditions $\underline{\underline{e_j}}(t_0)$, $\overline{\overline{e_j}}(t_0)$ will converge to zero. Hence, since the terms $\underline{\underline{e_j^w}}(\cdot)$, $\overline{\overline{e_j^w}}(\cdot)$ are always positive, if they are also sufficiently exciting (e.g., but not necessarily, $v(t)$ is a white noise), then $\underline{\underline{e_j}}(\cdot)$ and $\overline{\overline{e_j}}(\cdot)$ will eventually become positive and (46) will hold.

4. SIMULATION RESULTS

The aim of this section is to illustrate the comparison between local and glocal observers by means of a simulation study. Let us begin introducing the following system matrices:

$$A(\kappa) = A_0 + \kappa A_\bullet$$
$$B(\kappa) = B_0 + \kappa B_\bullet$$

with

$$A_0 = \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.3 \end{bmatrix} \quad B_0 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$A_\bullet = \begin{bmatrix} 0.2 & 0.1 \\ -0.1 & -0.3 \end{bmatrix} \quad B_\bullet = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

The unknown parameter κ is assumed to vary in the discrete set $\{-0.5,\ 0,\ 0.5\}$ according to the following switching scheme:

$$\kappa(t) := \begin{cases} -0.5 & t \le 100 \\ 0.5 & 100 \le t < 200 \\ 0 & t \ge 200 \end{cases}$$

Assuming that $C = I$, the local, global and glocal observer gains have been computed such that the corresponding closed-loop matrices equal the zero matrix. The bounds for local, global and glocal observers are computed based on (17)-(18), (34)-(35) and (47)-(48), respectively, assuming that each element of the state and input matrices is affected by bounded uncertainty between -0.05 and 0.05. The results obtained for an input $u(t) = -3 + \cos(0.06t)$ from an initial condition $[0, 0]^T \leq x(0) \leq [0.5, 0.5]^T$, are depicted in Figs. 1-4. It is worth to note that, as expected, the glocal bounds are more conservative than the local ones, but the latter are more likely to be violated due to changes in the system's active mode.

In a second simulation study, a statistical analysis has been performed to compare the estimation effectiveness of local and glocal observers. A family of 1000 switched systems with a fixed structure has been generated by random coefficient selection. The system matrices have been designed to depend linearly on the switching parameter $\kappa(t)$, whose range is constrained to a discrete set containing three admissible values. In the simulations the parameter $\kappa(t)$ is forced to undergo a switch at $t = 200$ and at $t = 600$. When a switch occurs, a transient is triggered both in the local and the glocal observers, with a consequent possible temporary violation of bounds (17)-(18) and (47)-(48). Focusing on the time samples following the switches, we are interested in comparing the number of systems that are violating the bounds. The two scenarios are represented in the histograms in Figs. 5-6. In both cases, it is clear that a smaller proportion of glocal observers fails to remain within the bounds compared to local observers and, furthermore, that the rate of recovery for glocal observers is remarkably higher. In conclusion, as already mentioned, on the one hand glocal observers introduce a larger conservativeness for the computation of bounds but, on the other hand, they also guarantee a better accuracy for the identification of correct switching modes.

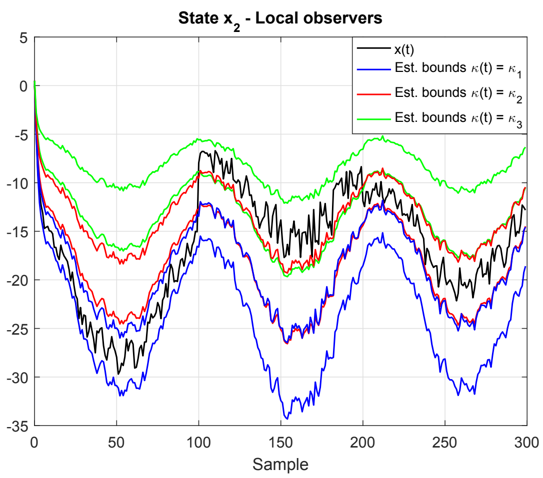

Fig. 1. Evolution of the state x_2: estimation using a bank of local interval observers

5. CONCLUSIONS

This paper has analyzed how different types of interval observers behave after a commutation of active mode in uncertain discrete-time switched systems, and how the effects of non-positivity of interval state estimation errors can be reduced. In particular, the performance of local interval observers has been compared to the one obtained using glocal interval observers, i.e. local observers fed by the information provided by a global interval observer. The simulation results show that, although glocal observers introduce conservativeness in the computation of the interval bounds, they also guarantee a better accuracy for the identification of correct switching modes.

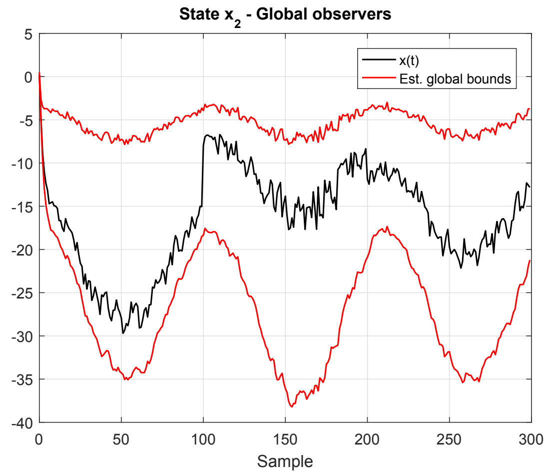

Fig. 2. Evolution of the state x_2: estimation using a global observer

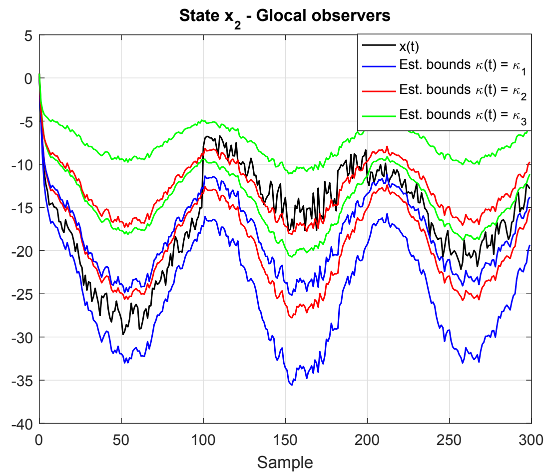

Fig. 3. Evolution of the state x_2: estimation using a bank of glocal interval observers

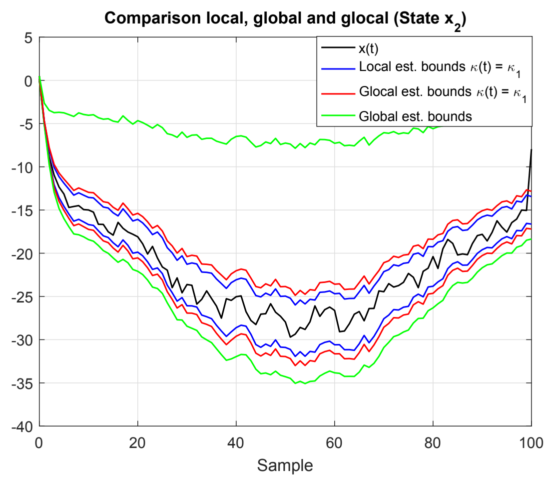

Fig. 4. Evolution of the state x_2: comparative analysis of the local, global and glocal observers in mode κ_1.

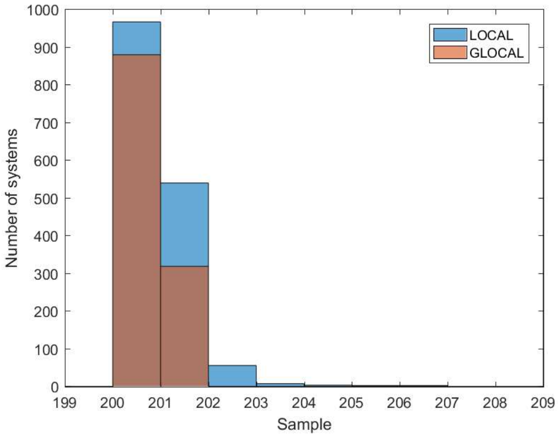

Fig. 5. Statistical analysis of bound violation due to the switch at $t = 200$

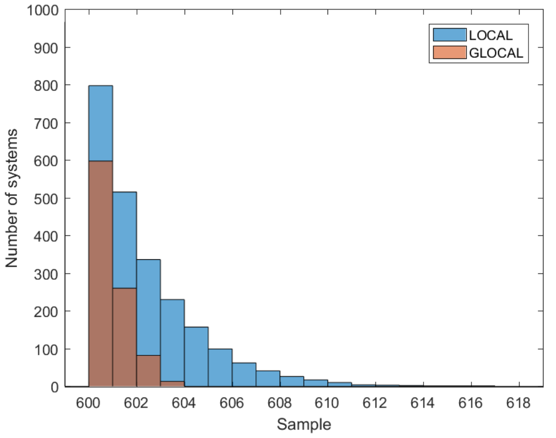

Fig. 6. Statistical analysis of bound violation due to the switch at $t = 600$

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